MATH 54 − HINTS TO HOMEWORK 11

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Here are a couple of hints to Homework 11! Enjoy :)

SECTION 6.5: LEAST SQUARES PROBLEMS

Here's the general procedure to solve least-squares problems:

To find the least-squares solution of $A\mathbf{x} = \mathbf{b}$ in the least-squares sense, multiply both sides by A^T, to get the **normal equations**:

$$A^T A\widetilde{\mathbf{x}} = A^T \mathbf{b}$$

Your solution $\widetilde{\mathbf{x}}$ is called the least-squares solution and he least squares error is $\|A\widetilde{\mathbf{x}} - \mathbf{b}\|$.

6.5.9. You have to do this directly, and notice this works because the columns of A are **orthogonal**: If you call the columns of A $\mathbf{u}$ and $\mathbf{v}$ respectively, then:

$$\hat{\mathbf{b}} = \left(\frac{\mathbf{b}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} + \left(\frac{\mathbf{b}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$$

Then solve $A\widetilde{\mathbf{x}} = \hat{\mathbf{b}}$

6.5.13. No! Because $\|A\mathbf{v} - \mathbf{b}\|$ is **smaller** than $\|A\mathbf{u} - \mathbf{b}\|$, so $\mathbf{u}$ **cannot** be a least-squares solution of $A\mathbf{x} = \mathbf{b}$, by the **definition** of a least-squares solution! (beginning of section 6.5)

6.5.17.

(a) **T**
(b) **T**
(c) **F** (it's the reverse)
(d) **T**
(e) **T** (this is the uniqueness-test I talked about in lecture)

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6.5.18.

(a) **T** (because if $\mathbf{b}$ is in $Col(A)$, then $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$, which is also the least-squares solution!)
(b) **F** (it's the $\mathbf{x}$ such that $A\mathbf{x}$ is closest to $\mathbf{b}$, the least-squares solution does **not** lie in $Col(A)$, draw a picture to convince yourself of that!)
(c) **T** (It's a fancy way of saying: Find $\widetilde{\mathbf{x}}$ such that $A\widetilde{\mathbf{x}} = \hat{\mathbf{b}}$
(d) **F** (AA^T might not be invertible! But if it is, then yes)

6.5.25. It's the same as usual, except that there will be many least squares solutions! You should find $x + y = 3$

Section 6.6: Applications to linear models

You can **IGNORE** this section if you want!

6.6.3. Suppose $y = \beta_0 + \beta_1 x$, then assuming the points are on the line, you get:

$$\begin{cases} \beta_0 - (1)\beta_1 = & 0 \\ \beta_0 + (0)\beta_1 = & 1 \\ \beta_0 + (1)\beta_1 = & 2 \\ \beta_0 + (2)\beta_1 = & 4 \end{cases}$$

That is:

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

Now solve this using least-squares!

Section 6.7: Inner product spaces

Note: In this section, I would like you to actually evaluate any integrals you find! No cheating and using your calculator :) The reason is that 1) there are only 3 problems where you have to do this and 2) On the exam, you're not allowed to use a calculator!

6.7.15. Here $\overset{(1)}{=}$ means: We use axiom 1 on page 368, etc.

$$\langle \mathbf{u}, c\mathbf{v} \rangle \overset{(1)}{=} \langle c\mathbf{v}, \mathbf{u} \rangle \overset{(2)}{=} c \langle \mathbf{v}, \mathbf{u} \rangle \overset{(1)}{=} c \langle \mathbf{u}, \mathbf{v} \rangle$$

6.7.17. Write $\|\mathbf{u} + \mathbf{v}\|^2$ as $\langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle$ and expand everything out. Ditto for the other term!

6.7.19, 6.7.20. The Cauchy-Schwarz inequality says $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$. For 6.7.20, square each term!

6.7.21, 6.7.23. $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$. Also, $\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_0^1 (f(t))^2 \, dt}$

6.7.25, 6.7.26. Use the usual Gram-Schmidt process, except with $\mathbf{u_1} = 1$, $\mathbf{u_2} = t$, $\mathbf{u_3} = t^2$, and $\mathbf{f} \cdot \mathbf{g} = \int_{-1}^{1} f(t)g(t)dt$ for 6.7.24 and $\mathbf{f} \cdot \mathbf{g} = \int_{-2}^{2} f(t)g(t)dt$. If you don't remember the Gram-Schmidt formulas, here they are:

$\mathbf{v_1} = \mathbf{u_1}$ and cross out $\mathbf{u_1}$ from your list!

Then calculate $\hat{\mathbf{u_2}} = \left(\frac{\mathbf{u_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right) \mathbf{v_1}$.

Then let $\mathbf{v_2} = \mathbf{u_2} - \hat{\mathbf{u_2}}$, and cross out $\mathbf{u_2}$ from your list!

Then calculate:
$\hat{\mathbf{u_3}} = \left(\frac{\mathbf{u_3} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right) \mathbf{v_1} + \left(\frac{\mathbf{u_3} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}}\right) \mathbf{v_2}$

Then let $\mathbf{v_3} = \mathbf{u_3} - \hat{\mathbf{u_3}}$, and cross out $\mathbf{u_3}$ from your list!

Evaluate any integrals that you find! Don't just copy the answer from the back of the book!